TRINITY COLLEGE

Semester One Examination, 2023

Question/Answer booklet

# MATHEMATICS SPECIALIST UNIT 1

SOLUTIONS

## Section One: Calculator-free

WA student number: In figures

In words

Your name

## Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

## Materials required/recommended for this section

***To be provided by the supervisor***

This Question/Answer booklet
Formula sheet

***To be provided by the candidate***

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

## Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

## Structure of this paper

| Section | Number of questions available | Number of questions to be answered | Working time (minutes) | Marks available | Percentage of examination |
|---|---|---|---|---|---|
| Section One: Calculator-free | 7 | 7 | 50 | 49 | 35 |
| Section Two: Calculator-assumed | 12 | 12 | 100 | 94 | 65 |
| | | | | **Total** | 100 |

## Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

**Section One: Calculator-free** **35% (49 Marks)**

This section has **seven** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

---

**Question 1** **(6 marks)**

Relative to an origin $O$, the position vectors of points $K, L$ and $M$ are $\begin{pmatrix}5\\1\end{pmatrix}, \begin{pmatrix}-1\\-7\end{pmatrix}$ and $\begin{pmatrix}-3\\-1\end{pmatrix}$ respectively.

(a) Determine $\overrightarrow{LK}$. (1 mark)

| Solution |
|---|
| $\overrightarrow{LK} = \begin{pmatrix}5\\1\end{pmatrix} - \begin{pmatrix}-1\\-7\end{pmatrix} = \begin{pmatrix}6\\8\end{pmatrix}$ |
| **Specific behaviours** |
| ✓ correct vector |

(b) Determine the magnitude of $\overrightarrow{LK}$. (1 mark)

| Solution |
|---|
| $\lvert\overrightarrow{LK}\rvert = \sqrt{36+64} = 10$ |
| **Specific behaviours** |
| ✓ correct magnitude |

(c) Determine a vector in the same direction as $\overrightarrow{LM}$ that has the same magnitude as $\overrightarrow{LK}$. (2 marks)

| Solution |
|---|
| $\overrightarrow{LM} = \begin{pmatrix}-3\\-1\end{pmatrix} - \begin{pmatrix}-1\\-7\end{pmatrix} = \begin{pmatrix}-2\\6\end{pmatrix}, \quad \lvert\overrightarrow{LM}\rvert = \sqrt{4+36} = 2\sqrt{10}$ <br> $\underset{\sim}{r} = \dfrac{10}{2\sqrt{10}}\begin{pmatrix}-2\\6\end{pmatrix} = \sqrt{10}\begin{pmatrix}-1\\3\end{pmatrix}$ |
| **Specific behaviours** |
| ✓ correct direction <br> ✓ correct magnitude |

(d) If $KM$ is a diagonal of parallelogram $KLMN$, determine the position vector of vertex $N$. (2 marks)

| Solution |
|---|
| $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$, but since parallelogram $\overrightarrow{MN} = \overrightarrow{LK}$. <br> $\overrightarrow{ON} = \begin{pmatrix}-3\\-1\end{pmatrix} + \begin{pmatrix}6\\8\end{pmatrix} = \begin{pmatrix}3\\7\end{pmatrix}$ |
| **Specific behaviours** |
| ✓ indicates correct method <br> ✓ correct position vector |

**Question 2** **(6 marks)**

A student, mindful of the fact that a square, rectangle and rhombus are special cases of a parallelogram, wrote the following true statement:

If a quadrilateral is a parallelogram, then opposite sides in the quadrilateral are equal in length.

(a) Write the inverse statement and state whether it is true. (2 marks)

| Solution |
|---|
| If a quadrilateral is not a parallelogram, then opposite sides in the quadrilateral are not equal in length.<br><br>Inverse statement is false (e.g., isosceles trapezium). |
| **Specific behaviours** |
| ✓ correct inverse statement<br>✓ states false |

(b) Write the contrapositive statement and state whether it is true. (2 marks)

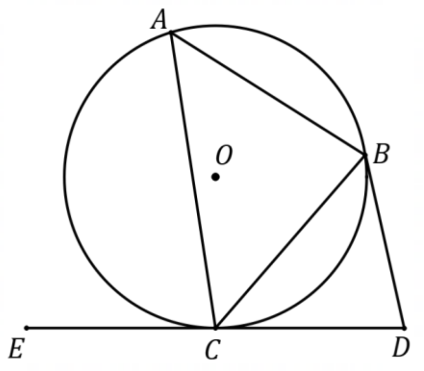
| Solution |
|---|
| If opposite sides in a quadrilateral are not equal in length, then the quadrilateral is not a parallelogram.<br><br>Contrapositive statement is true. |
| **Specific behaviours** |
| ✓ correct contrapositive statement<br>✓ states true |

(c) Write the converse statement and state whether it is true. (2 marks)

| Solution |
|---|
| If opposite sides in a quadrilateral are equal in length, then the quadrilateral is a parallelogram.<br><br>Converse statement is false (e.g., isosceles trapezium). |
| **Specific behaviours** |
| ✓ correct converse statement<br>✓ states false |

**Question 3** **(7 marks)**

In the diagram, $DE$ is tangential to the circle with centre $O$ at $C$ and the vertices of triangle $ABC$ lie on the circle as shown.




(a) Prove the angle in the alternate segment theorem, that $\angle BCD = \angle BAC$. (4 marks)

| Solution |
|---|
| Let $\angle BCD = x$.<br><br>$\angle BCO = 90° - x$ as radius $OC$ is perpendicular to tangent $DE$.<br><br>$\angle BOC = 180° - 2(90° - x) = 2x$ as $\Delta OBC$ is isosceles as $OB, OC$ are radii.<br><br>$\angle BAC = \frac{1}{2}\angle BOC = \frac{1}{2}(2x) = x$ as angle at centre is twice angle on circumference.<br><br>Hence $\angle BCD = \angle BAC = x$ as required.<br><br>*Note that several alternative proofs exist.* |
| **Specific behaviours** |
| ✓ expression for $\angle BCO$<br>✓ expression for $\angle BOC$<br>✓ expression for $\angle BAC$<br>✓ completes proof and supplies adequate explanation throughout |

(b) If $\angle CDB = \alpha, \angle CBD = \beta$ and $CA = CB$, prove that $\angle ACB = 2\alpha + 2\beta - 180°$. (3 marks)

| Solution |
|---|
| $\angle BCD = 180° - \alpha - \beta$ as angle sum of $\Delta CBD = 180°$.<br><br>$\angle CAB = \angle BCD = 180° - \alpha - \beta$ using proof from part (a)<br><br>$\angle ACB = 180° - 2\angle CAB$ as $\Delta CAB$ is isosceles with $CA = CB$.<br>$\angle ACB = 180° - 2(180° - \alpha - \beta) = 2\alpha + 2\beta - 180°$, as required. |
| **Specific behaviours** |
| ✓ expression for $\angle BCD$ with reasons<br>✓ expression for $\angle CAB$ with reasons<br>✓ expression for $\angle ACB$ with reasons |

**Question 4** **(9 marks)**

Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$ and $\underset{\sim}{c} = \begin{pmatrix} 9 \\ -8 \end{pmatrix}$. Determine

(a) a unit vector in the same direction as $3\underset{\sim}{a} - \underset{\sim}{c}$. (2 marks)

| Solution |
|---|
| $3\underset{\sim}{a} - \underset{\sim}{c} = 3\begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 9 \\ -8 \end{pmatrix} = \begin{pmatrix} 0 \\ 11 \end{pmatrix} \Rightarrow \underset{\sim}{\hat{u}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ correct sum<br>✓ correct unit vector |

(b) vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ given that $\underset{\sim}{a} = \underset{\sim}{r} + \underset{\sim}{s}$ and $\underset{\sim}{b} = 2\underset{\sim}{r} + \underset{\sim}{s}$. (3 marks)

| Solution |
|---|
| Subtracting first from second gives $\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{s}$ and so<br>$\underset{\sim}{s} = \begin{pmatrix} 5 \\ -2 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$<br>$\underset{\sim}{r} = \underset{\sim}{a} - \underset{\sim}{s} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ |
| **Specific behaviours** |
| ✓ uses equations to eliminate one unknown vector<br>✓ obtains one vector<br>✓ obtains second vector |

(c) the value of the constants $\lambda$ and $\mu$ when $\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = \underset{\sim}{c}$. (4 marks)

| Solution |
|---|
| $\lambda\begin{pmatrix} 3 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -8 \end{pmatrix} \Rightarrow 3\lambda + 5\mu = 9, \quad \lambda - 2\mu = -8$<br>$3\lambda + 5\mu = 9$<br>$3\lambda - 6\mu = -24$<br>$11\mu = 33$<br>$\mu = 3$<br>$\lambda = -8 + 2(3) = -2$<br>Hence $\lambda = -2, \mu = 3$. |
| **Specific behaviours** |
| ✓ equation using $\underset{\sim}{i}$ coefficients<br>✓ equation using $\underset{\sim}{j}$ coefficients<br>✓ solves equation for one value<br>✓ correct values |

**Question 5** **(7 marks)**

(a) Consider parallelogram $OABC$, so that $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$. Use a vector method to show that when diagonals $\overrightarrow{OB}$ and $\overrightarrow{AC}$ are perpendicular, then the parallelogram is a rhombus.

(4 marks)

**Solution**

$$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}, \qquad \overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$$

Vectors are perpendicular:

$$\left(\underset{\sim}{a} + \underset{\sim}{c}\right) \cdot \left(\underset{\sim}{c} - \underset{\sim}{a}\right) = 0$$

$$\underset{\sim}{a} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$$

$$\underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$$

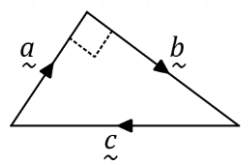
$$\left|\underset{\sim}{c}\right|^2 = \left|\underset{\sim}{a}\right|^2$$

Hence lengths of all sides of $OABC$ are equal and it is a rhombus.

**Specific behaviours**

✓ states vectors for both diagonals
✓ uses perpendicularity to form scalar product
✓ expands and simplifies scalar product
✓ uses equal magnitudes to deduce shape is a rhombus

(b) Vectors $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{c}$ are chosen as in the diagram shown, so that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.
Use a vector method to prove Pythagoras' theorem.




(3 marks)

**Solution**

From the diagram, $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} = \underset{\sim}{0}$ and so $\underset{\sim}{a} + \underset{\sim}{b} = -\underset{\sim}{c}$.

Hence

$$\left(\underset{\sim}{a} + \underset{\sim}{b}\right) \cdot \left(\underset{\sim}{a} + \underset{\sim}{b}\right) = \left(-\underset{\sim}{c}\right) \cdot \left(-\underset{\sim}{c}\right)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{c} \cdot \underset{\sim}{c}$$

But $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular and so $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a} = 0$.

Hence $\underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} = \underset{\sim}{c} \cdot \underset{\sim}{c}$ and so $\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2 = \left|\underset{\sim}{c}\right|^2$, as required.

**Specific behaviours**

✓ uses triangle to relate the three vectors
✓ equates dot products and expands
✓ simplifies to obtain required result

**Question 6** **(7 marks)**

(a) The letters of the word MATAMATA are arranged randomly in a line. Determine

(i) the number of different ways this can be done. (2 marks)

| Solution |
|---|
| 8 letters, $4\times$A, $2\times$M, $2\times$T. Number of arrangements is: $$\frac{8!}{4!\,2!\,2!}=\frac{8\times7\times6\times5}{2\times2}=10\times42=420$$ |
| **Specific behaviours** |
| ✓ correct expression using factorials<br>✓ correct number of arrangements |

(ii) the number of arrangements in which all the A's are grouped together. (2 marks)

| Solution |
|---|
| Let AAAA be one object (with one arrangement). Then arrange this object, $2\times$M, $2\times$T. Number of arrangements is: $$\frac{5!}{2!\,2!}=\frac{5\times4\times3\times2}{2\times2}=30$$ |
| **Specific behaviours** |
| ✓ correct expression using factorials<br>✓ correct number of arrangements |

(b) A class of 13 children together picked up 75 pieces of litter. Prove that at least two children picked up the same number of pieces of litter. (3 marks)

| Solution |
|---|
| Assume all children picked up a different number of pieces of litter. The fewest pieces of litter collected would be when 13 pigeonholes (labelled $0, 1, 2, \ldots 12$) were each filled once, a total of $12\times6.5=78$ pieces of litter. However, only 75 pieces were picked up, which contradicts our assumption, and so at least one pair of children picked up the same number of pieces. |
| **Specific behaviours** |
| ✓ states assumption<br>✓ uses assumption to determine minimum pieces<br>✓ explains contradiction to complete proof |

**Question 7** **(7 marks)**

(a) Vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ have magnitudes 4 and 3 respectively, and $\underset{\sim}{r} \cdot \underset{\sim}{s} = -2$. Evaluate

(i) $\left(2\underset{\sim}{r}\right) \cdot \left(\frac{1}{4}\underset{\sim}{s}\right)$. (1 mark)

| Solution |
|---|
| $\left(2\underset{\sim}{r}\right) \cdot \left(\frac{1}{4}\underset{\sim}{s}\right) = \frac{1}{2}\left(\underset{\sim}{r} \cdot \underset{\sim}{s}\right) = \frac{1}{2}(-2) = -1$ |
| **Specific behaviours** |
| ✓ correct value |

(ii) $\left(3\underset{\sim}{r} - \underset{\sim}{s}\right) \cdot \left(\underset{\sim}{r} - 4\underset{\sim}{s}\right)$. (3 marks)

| Solution |
|---|
| $\left(3\underset{\sim}{r} - \underset{\sim}{s}\right) \cdot \left(\underset{\sim}{r} - 4\underset{\sim}{s}\right) = 3\underset{\sim}{r} \cdot \underset{\sim}{r} - \underset{\sim}{s} \cdot \underset{\sim}{r} - 12\underset{\sim}{r} \cdot \underset{\sim}{s} + 4\underset{\sim}{s} \cdot \underset{\sim}{s}$<br>$= 3\left\lvert\underset{\sim}{r}\right\rvert^2 + 4\left\lvert\underset{\sim}{s}\right\rvert^2 - 13\underset{\sim}{r} \cdot \underset{\sim}{s}$<br>$= 3(4)^2 + 4(3)^2 - 13(-2)$<br>$= 48 + 36 + 26 = 112$ |
| **Specific behaviours** |
| ✓ correctly expands using scalar products<br>✓ simplifies product using $\underset{\sim}{n} \cdot \underset{\sim}{n} = \left\lvert\underset{\sim}{n}\right\rvert^2$<br>✓ correct value |

(b) The vector projection of $\begin{pmatrix} 3 \\ t \end{pmatrix}$ on $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 6 \\ -2 \end{pmatrix}$. Determine the value of the constant $t$. (3 marks)

| Solution |
|---|
| Vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b}$, and so<br>$\frac{\begin{pmatrix} 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$<br>$\frac{\begin{pmatrix} 3 \\ t \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}}{\begin{pmatrix} -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \end{pmatrix}} = -2$<br>$\frac{t - 9}{10} = -2$<br>$t - 9 = -20$<br>$t = -11$ |
| **Specific behaviours** |
| ✓ forms equation using vector projection<br>✓ correctly evaluates scalar products<br>✓ correct value |



**End of questions**

Supplementary page

Question number: _________

Supplementary page

Question number: _________